Solution to Problems ♠–13

Problem A: *Let a_n be the number written with 2^n nines. For example, $a_0 = 9$, $a_1 = 99$, $a_2 = 9999$. Let $b_n = \prod_{i=0}^{n} a_i$. Find the sum of the digits of b_n.*

Answer: Answer: $9 \cdot 2^n$.

We prove this by induction on $n \in \mathbb{N}$. Let $P(n)$ be the statement that "the sum odf digits of b_n is $9 \cdot 2^n$". We are going to verify that $P(n)$ satisfies the assumptions of the Theorem on Mathematical Induction.

Basic Step: We have $b_0 = 9$, digit sum 9, and $b_1 = 891$, digit sum 18, so the result is true for $n = 0$ and $n = 1$.

Inductive Step: Let $n \in \mathbb{N}$ be arbitrary and let us assume $P(n-1)$ is true. Obviously $a_n < 10^{2^n}$, so

$$b_{n-1} < 10^{1+2+2^2+\ldots+2^{n-1}} < 10^{2^n}.$$

Now $b_n = b_{n-1}10^{2^n} - b_{n-1}$. But $b_{n-1} < 10^{2^n}$, so

$$b_n = (b_{n-1} - 1)10^{2^n} + (10^{2^n} - b_{n-1})$$

and the digit sum of b_n is just the digit sum of $(b_{n-1} - 1) \cdot 10^{2^n}$ plus the digit sum of $10^{2^n} - b_{n-1}$.

Now b_{n-1} is odd and so its last digit is non-zero, so the digit sum of $b_{n-1} - 1$ is one less than the digit sum of b_{n-1}, and hence is $9 \cdot 2^{n-1} - 1$. Multiplying by 10^{2^n} does not change the digit sum. $(10^{2^n} - 1) - b_{n-1}$ has 2^n digits, each 9 minus the corresponding digit of b_{n-1}, so its digit sum is $9 \cdot 2^n - 9 \cdot 2^{n-1}$. Since b_{n-1} is odd, its last digit is not 0 and hence the last digit of $(10^{2^n} - 1) - b_{n-1}$ is not 9. So the digit sum of $10^{2^n} - b_{n-1}$ is $9 \cdot 2^n - 9 \cdot 2^{n-1} + 1$. Hence b^n has digit sum

$$(9 \cdot 2^{n-1} - 1) + (9 \cdot 2^n - 9 \cdot 2^{n-1} + 1) = 9 \cdot 2^n.$$

Consequently, by the Theorem on Mathematical Induction we may conclude that $(\forall n \in \mathbb{N})P(n)$, i.e., our assertion is true indeed.

No Correct solutions received from

Problem B: *Show that for each natural number n,*

$$\sum_{i=1}^{n} \frac{1}{i} \leq 1 + \ln(n).$$

Answer: We show this by induction on $n \in \mathbb{N}$. Let $\Phi(n)$ be the statement "$\sum\limits_{i=1}^{n} \frac{1}{i} \leq 1 + \ln(n)$". We will verify that $\Phi(n)$ satisfies the assumptions of the Theorem on Mathematical Induction.

Basic Step: We note that $\Phi(1)$ is the assertion that "$\sum\limits_{i=1}^{1} \frac{1}{i} \leq 1 + \ln(1)$", but this is readily true.

Inductive Step: Let $n \in \mathbb{N}$ be arbitrary and let us argue tha th eimplication $\Phi(n) \Rightarrow \Phi(n+1)$ holds true.

So suppose $\Phi(n)$ is true, that is

$(\otimes)_n$ $\sum\limits_{i=1}^{n} \frac{1}{i} \leq 1 + \ln(n)$.

Now note that

$$\sum_{i=1}^{n+1} \frac{1}{i} = \frac{1}{n+1} + \sum_{i=1}^{n} \frac{1}{i} \leq_{\text{by } (\otimes)_n} \frac{1}{n+1} + 1 + \ln(n).$$

Note that by the Lagrange's Mean Value Theorem, for some $x \in (n, n+1)$ we have

$$\ln(n+1) - \ln(n) = \frac{1}{x} \geq \frac{1}{n+1},$$

and hence

$$\frac{1}{n+1} + 1 + \ln(n) \leq 1 + \ln(n+1).$$

Consequently, under the assumption $(\otimes)_n$, we have

$$\sum_{i=1}^{n+1} \frac{1}{i} \leq 1 + \ln(n+1).$$

By the Theorem on Mathematical Induction we may conclude that $(\forall n \in \mathbb{N})\Phi(n)$, i.e., our claim is true indeed.

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